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EQUATIONS OF MOTION FOR TWO-PHASE FLOW IN A PIN BUNDLE OF A NUCLEAR REACTOR

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Abstract—By performing Eulerian area averaging over a channel area of the local continuity, momentum and energy equations for single phase turbulent flow, assuming each phase in two-phase flows to be continuum but coupled by the appropriate "jump" conditions at the interface, the corresponding axial macroscopic balances for two-fluid model in a pin bundle are obtained. To determine the crossflow, a momentum equation in transverse (to the gap between the pins) direction is obtained for each phase by carrying out Eulerian segment averaging of the local momentum equation, where the segment is taken parallel to the gap. By considering the mixture as a whole, a diffusion model based on drift-flux velocity is formulated. In axial direction it is expressed in terms of three mixture conservation equations of mass, momentum and energy with one additional continuity equation for the vapor phase. For the determination of crossflow, transverse momentum equation for a mixture is obtained. It is discussed that the previous formulation of the two-phase flow based on the "slip" flow model and integral subchannel balances using finite control volumes is inadequate in the following respects:

(a) The model is heuristic and *a priori* assumes the order of magnitude of the terms. An excellent example of this provided by the manner in which the form of transverse momentum equation has evolved since its first crude form.

(b) The model is incomplete and incorrect when applied to two-phase mixtures in thermal non-equilibrium such as during accidental depressurization of a water cooled reactor.

It is demonstrated that the governing equations presented here are based on very formal and sound physical basis and are indispensable if physically correct methods are desired for analyzing two-phase flows in a pin bundle.

NOMENCLATURE

- A, cross-sectional area normal to z axis;
- C_k, correlation coefficient for momentum defined by equation (33) for the kth phase;
- c, area-weighted mass concentration defined by equation (17);
- E, specific total internal energy;
- $\mathbf{g}$, acceleration due to gravity;
- H, specific static enthalpy;
- j, velocity of center of volume of mixture;
- $\mathbf{K}$, unit vector in z direction;
- L, segment parallel to y direction;
- $\dot{m}_k$, mass flux at the interface;
- N_i, the number of channels connected to a channel i;
- P, pressure;
- $\mathbf{q}$, heat flux;
- S, surface area of a bounding volume;
- s, perimeter along a boundary;
- t, time;
- U_k, specific internal energy for the kth phase;
- $\mathbf{V}$, fluid velocity vector;
- W_{kil}, cross flow of the kth phase per unit axial length between channels i and l;
- (x, y, z), coordinate system.

Greek symbols

- α, void fraction;
- Γ_k, mass produced of the kth phase per unit volume of the mixture;
- γ_k, correlation coefficient for energy defined by equation (46b) for the kth phase;
- ρ, fluid density;
- τ, stress tensor.

Subscripts

- e, boundary between the interconnected channels;
- fg, phase change between liquid to vapor;
- I, vapor–liquid interface;
- i, channel i;
- k, kth phase;
- m, mixture;
- t, tangential direction;
- z, directed along z axis.

INTRODUCTION

IN THE thermal hydraulic analysis of pin bundles for thermal or fast reactors, it has become customary to employ heuristic macroscopic balances using finite control volumes (e.g. subchannels) for mass, momentum and energy (see for example [1–4]). However, a comparison between the experimental data and the subchannel calculation based on these models, several discrepancies in the prediction of mass flow and enthalpy distribution compared to experimental data have been noted (see for example [5–8]). In order to establish the physical basis and thus the validity and to reveal the terms not accounted for in these models, we are lead to the derivation of the governing equation based on macroscopic balances of mass, momentum and energy applicable to a pin bundle. These in turn are derived from differential balances (as for example expressed by Reynolds equations for turbulent flow) by utilizing suitable averaging procedures such as Eulerian area and segment averaging.

The knowledge of the proper governing equation for two-phase flows in a pin bundle such as may occur during postulated loss-of-flow accident in thermal and fast reactors, is even more lacking than the single-phase flows due to added complexity of two-phase flows. A number of computer codes [1–4] for multi-channel two-phase thermal hydraulic calculations have been developed. Some of these have been specifically designed for application to water reactors. The governing equations in these codes are obtained heuristically by assuming a complete analogy with the single phase flows. The methodology adopted in this paper for deriving the governing equation for two-phase flows is formal and basic and does not *a priori* assume the order of magnitude of the terms as would be necessary for macroscopic balances using finite control volumes.

It is well established in continuum mechanics that the conceptual models for single phase flow of a gas or of a liquid are formulated in terms of differential field equations which are based on the laws of conservation of mass, momentum and energy. These field equations are then complemented by appropriate constitutive equations such as equations of state. It is expected, therefore, that the conceptual models which describe the steady state and dynamic characteristic of two-phase flows should be formulated in terms of appropriate field and constitutive equations. From the mathematical point of view, a two-phase flow can be considered as a field which is subdivided into two single phase regions with moving boundaries separating the two constituent phases. The differential balance holds for each subregion, however, it cannot be applied to these sub-regions in the normal sense without accounting for the presence of interfaces. The presence of interfaces must be described in terms of macroscopic balances applied at the interfaces taking into account the singular characteristics, i.e. the sharp changes (or discontinuities) in various variables. The governing equations for two-phase flow in a pin bundle must then be obtained by employing Eulerian area and a segment averaging of these field equations. This procedure if properly carried out should establish a more firm basis for the governing equations and should reveal the additional terms not accounted for in the heuristic approach as adopted in the existing subchannel models. This is indeed the objective of the present paper.

The greater part of the formal procedure adopted here for the derivation of governing equations for a pin bundle is based on the methodology developed by Kocamustafaogullari [9] for obtaining one-dimensional macroscopic balances for a control volume. Kocamustafaogullari, however started with local differential equations for laminar flow whereas our fundamental equations are those for turbulent flow. In addition, we have included the derivation of the macroscopic transverse momentum balance for the calculation of cross flow in a pin bundle geometry.

BASIC EQUATIONS

In analyzing two-phase flows, we would follow the standard method of continuum mechanics by considering a two-phase flow as a field subdivided into two single phase regions with moving boundaries between the phases. The basic conservation equations in differential form hold for each subregion with appropriate jump and boundary conditions for matching the solution of these differential equations at the interfaces. From these field equations, we obtain macroscopic description by performing area and segment averaging by using a control volume which includes both phases at the same moment but is assumed to be much smaller than the total system of interest. The local field equations for each phase by considering each phase to be unsteady, compressible and turbulent, can be given as:

Continuity

$$\frac{\partial \bar{\rho}_k}{\partial t} + \nabla \cdot \bar{\rho}_k \tilde{\mathbf{V}}_k = 0. \tag{1}$$

Momentum

$$\frac{\partial}{\partial t}(\bar{\rho}_k \tilde{\mathbf{V}}_k) + \nabla \cdot (\bar{\rho}_k \tilde{\mathbf{V}}_k \tilde{\mathbf{V}}_k) = -\nabla \bar{P}_k + \bar{\rho}_k \mathbf{g} + \nabla \cdot (\bar{\tau}_k - \overline{\rho \mathbf{V}'_k \mathbf{V}'_k}) \tag{2}$$

Energy

$$\frac{\partial}{\partial t}(\bar{\rho}_k \tilde{E}_k) + \nabla \cdot (\bar{\rho}_k \tilde{E}_k \tilde{\mathbf{V}}_k) = \nabla \cdot (-\bar{\mathbf{q}}_k - \overline{P_k \mathbf{V}_k} - \overline{\rho_k E'_k \mathbf{V}'_k} + \overline{\tau_k \cdot \mathbf{V}_k}) + \bar{\rho}_k \mathbf{g} \cdot \tilde{\mathbf{V}}_k \tag{3}$$

where ρ_k is the density [of kth phase, $k = 1$ (liquid), $k = 2$ (gas)], $\mathbf{V}_k$ is the velocity vector, P_k is the pressure, E_k is the total internal energy ($E_k = U_k + \frac{1}{2}V_k^2$, where U_k is the specific internal energy) τ_k is the stress tensor, $\mathbf{g}$ is acceleration due to gravity and t is the time. The bar over any quantity denotes the conventional time averaging or statistical ensemble averaging. Thus,

$$\bar{F} = (1/\Delta t)\int_{t_0}^{t_0+\Delta t} F(t)\,dt \tag{4a}$$

and the tilde over any quantity F denotes the mass-weighted averaging i.e.,

$$\tilde{F} = \frac{\overline{\rho F}}{\bar{\rho}}. \tag{4b}$$

MACROSCOPIC BALANCES

We assume that both in the liquid and in the vapor phases, the motion of the fluids is dominantly in axial direction, that is the transverse components of velocity are small compared to axial component. We further assume that within a channel the variation of the axial component V_{kz} is much stronger than the variation along the axial direction. This behavior is analogous to that which exists in a boundary layer so that the longitudinal or axial length scale L (in which the axial

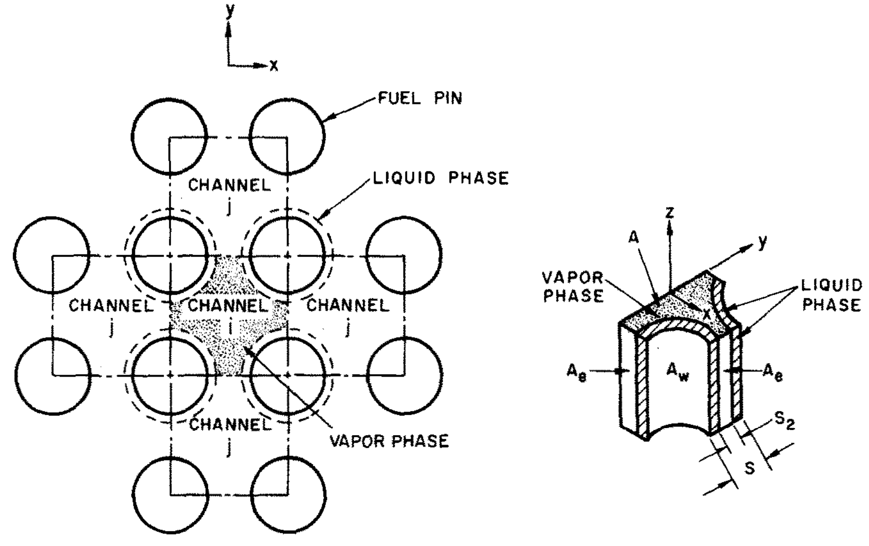

FIG. 1. Coordinate system in a subchannel.

variations in V_{kz} take place) is very much larger than the length scale δ in a transverse direction (over which transverse variations in V_{kz} take place) i.e. $\delta/L \ll 1$. We further assume that similar arguments can also be applied to the variation of enthalpy or temperature in a channel. In conclusion, it is assumed that the boundary-layer approximation can be applied.

Continuity equation averaged over a cross-section

The application of Leibnitz theorem (A1) and divergence theorem (A2) (see Appendix A for their statement) to equation (1) yields for a channel denoted by i in the subassembly (see Fig. 1) a mass balance given by

$$\frac{\partial}{\partial t}(A_k\langle\langle\bar{\rho}_k\rangle\rangle)_i+\frac{\partial}{\partial z}(A_k\langle\langle\bar{\rho}_k\tilde{V}_{kz}\rangle\rangle)_i = -\left[\int_{S_I}\bar{\rho}_k(\bar{\mathbf{V}}_k-\mathbf{V}_I)\cdot\hat{n}_k\frac{\mathrm{d}A_I}{\mathrm{d}z}\right]_i - \sum_{l=1}^{N_i}\left(\int_{S_k}\bar{\rho}_k\bar{\mathbf{V}}_k\cdot\hat{n}_k\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right)_{il}. \quad (5)$$

If we let

$$-\left[\frac{1}{A}\int_{S_I}\bar{\rho}_k(\bar{\mathbf{V}}_k-\mathbf{V}_I)\cdot\hat{n}_k\frac{\mathrm{d}A_I}{\mathrm{d}z}\right]_i = -\left(\frac{1}{A}\int_{S_I}\dot{m}_k\frac{\mathrm{d}A_I}{\mathrm{d}z}\right)_i = \Gamma_{ki} \quad (6a)$$

and

$$\left(\int_{S_k}\bar{\rho}_k\bar{\mathbf{V}}_k\cdot\hat{n}_k\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right)_{il} = W_{kil}. \quad (6b)$$

equation (5) becomes

$$\frac{\partial}{\partial t}(A_k\langle\langle\bar{\rho}_k\rangle\rangle)_i+\frac{\partial}{\partial z}(A_k\langle\langle\bar{\rho}_k\tilde{V}_{kz}\rangle\rangle)_i = (A\Gamma_k)_i-\sum_l W_{kil} \quad (7)$$

where summation with index l implies summation over the number N_i of channels connected to channel i. Here A_k is the cross-sectional area normal to z axis for phase k, and A denotes the total cross-sectional area normal to z, Γ_{ki} is the mass produced of the kth phase per unit volume of the mixture, and W_{kil} is the mass flow per unit axial length across the gap between channels i and l, and $(\langle\langle\ldots\rangle\rangle)$ defines the area averaged value of a quantity, i.e.

$$\langle\langle\Psi_k\rangle\rangle(z,t)=\frac{1}{A_k}\iint_{A_k}\Psi(x,y,z,t)\,\mathrm{d}A.$$

Adding equations for $k = 1$ and 2 and using equation (B1), we obtain

$$\frac{\partial}{\partial t}(A\rho_m)_i+\frac{\partial}{\partial z}(A\rho_m V_m)_i = -\sum_{k=1}^{2}\sum_l W_{kil} \quad (8)$$

where

$$V_m = [(1-\alpha)\langle\langle\bar{\rho}_1\tilde{V}_{1z}\rangle\rangle+\alpha\langle\langle\bar{\rho}_2\tilde{V}_{2z}\rangle\rangle]/\rho_m \quad (9a)$$

$$\rho_m = (1-\alpha)\langle\langle\bar{\rho}_1\rangle\rangle+\alpha\langle\langle\bar{\rho}_2\rangle\rangle. \quad (9b)$$

With a view of adopting well known and physically consistent approach as put forward in numerous works of Zuber and his co-workers (see for example [9–11]) towards the formulation of the governing equations for two-phase flows, we base the formulation of the governing equation for pin bundle on one-dimensional drift flux model for non-homogeneous, non-equilibrium flows of two-phase mixtures. The drift flux model (or mixture model) is formulated by considering the mixture as a whole, rather than two phases separately. The drift flux model thus requires only four field equations (namely, continuity, momentum, energy for the mixture, and the continuity equation for one of the phases say vapor) as against six field equations for two-fluid model. The use of two continuity equations is consistent with the traditional approach pursued in dealing with single phase two component chemically reacting flows in which the number of continuity equations utilized is equal to the number of components. The simplification introduction by using only four field equations rather than six makes the use of drift flux model a very attractive and powerful technique for analyzing a number of engineering problems such as voiding dynamics in a liquid metal fast breeder reactor (LMFBR) pin bundle or dynamics of two-phase flows in a pin bundle of a water-cooled reactor.

The four field equations in the drift flux model are the result of the elimination of one energy and one momentum equation from the original six field equations of the two-fluid model. Therefore, the relative motion and energy differences should be expressed by additional constitutive equations. These two effects inherent to all two-phase flow systems are taken into account by using a continuity equation for one of the phases and supplementing it with kinematic and phase change constitutive equations.

Many two-phase flow problems because of the importance of the relative motion of one phase with respect to the other must be formulated in terms of two velocity fields. For analyzing the dynamics of a two-phase mixture the appropriate velocity fields [10] are the velocity of center of mass as defined previously by equations (9) and the axial velocity of center of volume of the mixture

$$j = (1-\alpha)\langle V_{1z}\rangle + \alpha\langle V_{2z}\rangle. \tag{10}$$

Because of the existence of axial relative velocity between the phases given as

$$V_r = \langle\tilde{V}_{2z}\rangle - \langle\tilde{V}_{1z}\rangle, \tag{11}$$

the velocities V_m and j are not equal. Here $\langle\cdot\rangle$ denotes a mass weighted, area averaged quantity, i.e.

$$\langle\Psi_k\rangle = \frac{\langle\langle\bar{\rho}_k\Psi_k\rangle\rangle}{\langle\langle\bar{\rho}_k\rangle\rangle}. \tag{12}$$

The relationship between V_m and j is then obtained by the use of equations (9) through (11) as

$$j - V_m = \alpha(1-\alpha)\frac{\Delta\rho}{\rho_m}V_r. \tag{13}$$

which states that, the center of mass and the center of volume of the mixture move with different velocities and it is expected for example, in separated flow system that the center of mass will move with a velocity close to that of the heavy phase which accounts for most of the mass and center of volume velocity will be closer to that of the lighter phase which accounts for the most of the volume. The relationship between these two velocity fields can be expressed alternatively in terms of diffusion velocities of the phases with respect to the center of mass or the center of volume. The advantage of using these diffusion velocities as against relative velocity to express the relationship between these velocity fields, can easily be seen, for example in the case of dispersed two-phase flows in where the diffusion velocities can be easily related to a rise velocity of bubbles or a terminal velocity of particles or drops. Diffusion velocity V_{km} of the kth phase with respect to center of mass is defined as

$$V_{km} = \langle\tilde{V}_{kz}\rangle - V_m \quad \text{for} \quad k = 1,2 \tag{14}$$

the diffusion velocity V_{kj} with respect to center of volume, called drift velocity is defined as

$$V_{kj} = \langle\tilde{V}_{kz}\rangle - j. \tag{15}$$

Substituting for V_m from equation (9a) into equation (14), we obtain

$$\begin{aligned} V_{1m} &= -\frac{\langle\langle\bar{\rho}_2\rangle\rangle}{\rho_m}\alpha(\langle\tilde{V}_{2z}\rangle - \langle\tilde{V}_{1z}\rangle) \\ &= -\frac{\langle\langle\bar{\rho}_2\rangle\rangle}{\rho_m}\alpha V_r \end{aligned} \tag{16a}$$

$$\begin{aligned} V_{2m} &= \frac{\langle\langle\bar{\rho}_1\rangle\rangle}{\rho_m}(1-\alpha)(\langle\tilde{V}_{2z}\rangle - \langle\tilde{V}_{1z}\rangle) \\ &= \frac{\langle\langle\bar{\rho}_1\rangle\rangle}{\rho_m}(1-\alpha)V_r. \end{aligned} \tag{16b}$$

If we define area-weighted mass concentration c as

$$c(x,t) = \frac{\alpha\langle\langle\bar{\rho}_2\rangle\rangle}{\rho_m} \tag{17}$$

then equation (16a) can be written as

$$V_{1m} = -cV_r \tag{18a}$$

$$V_{2m} = (1-c)V_r \tag{18b}$$

equation (13) becomes

$$j - V_m = \alpha(1-\alpha)\frac{\Delta\rho}{\rho_m}(\tilde{V}_{2m} - \tilde{V}_{1m}) \tag{19}$$

and equation (9a) becomes

$$V_m = (1-c)\langle\tilde{V}_{1z}\rangle + c\langle\tilde{V}_{2z}\rangle. \tag{20}$$

Similarly by substituting for j from equation (10), we obtain

$$V_{1j} = -\alpha V_r \tag{21a}$$

$$V_{2j} = (1-\alpha)V_r \tag{21b}$$

$$V_r = V_{2j} - V_{1j}. \tag{21c}$$

The use of equation (21c) into equation (13) yields

$$j - V_m = \alpha(1-\alpha)\frac{\Delta\rho}{\rho_m}(V_{2j} - V_{1j}). \tag{22}$$

In view of equations (17), (18) and (21) the two types of diffusion velocities are related as

$$V_{1j} = \frac{\rho_m}{\langle\langle\bar{\rho}_1\rangle\rangle}V_{1m} = -\frac{\alpha}{1-\alpha}\frac{\rho_m}{\langle\langle\bar{\rho}_1\rangle\rangle}V_{2m} \tag{23}$$

$$V_{2j} - \frac{\rho_m}{\langle\langle\bar{\rho}_1\rangle\rangle}V_{2m} = -\frac{1-\alpha}{\alpha}\frac{\rho_m}{\langle\langle\bar{\rho}_2\rangle\rangle}V_{1m}. \tag{24}$$

By substituting equations (17) and (14) into equation (7) for vapor phase, we obtain

$$\frac{\partial}{\partial t}(Ac\rho_m)_i + \frac{\partial}{\partial z}(cA\rho_m V_m)_i = (A\Gamma_2)_i - \frac{\partial}{\partial z}(cA\rho_m V_{2m})_i - \sum_l W_{2il}. \tag{25}$$

Equations (8) and (25) constitute the required continuity equations for two-phase mixture flow.

Momentum equation averaged over area

The application of equations (A1) and (A2) to equation (2) yields

$$\frac{\partial}{\partial t}(A_k\langle\langle\bar{\rho}_k\rangle\rangle\langle\tilde{\mathbf{V}}_k\rangle)_i+\frac{\partial}{\partial z}(A_k\langle\langle\bar{\rho}_k\tilde{V}_{kz}\tilde{\mathbf{V}}_k\rangle\rangle)_i$$
$$+\left[\int_{s_I}\bar{\rho}_k(\tilde{\mathbf{V}}_k-\mathbf{V}_I)\cdot\hat{n}_k\tilde{\mathbf{V}}_k\frac{\mathrm{d}A_I}{\mathrm{d}z}\right]_i+\sum_l\left[\int_{s_k}\bar{\rho}_k(\tilde{\mathbf{V}}_k\cdot\hat{n}_k)\tilde{\mathbf{V}}_k\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right]_{il}$$
$$=\frac{\partial}{\partial z}[A_k\langle\langle(-\bar{\delta}\bar{P}_k+\bar{\tau}_k-\overline{\rho_k\mathbf{V}'_k\mathbf{V}'_k})\rangle\rangle\cdot\hat{K}]_i+\sum_l\left[\int_{s_k}(-\bar{\delta}\bar{P}_k+\bar{\tau}_k-\overline{\rho_k\mathbf{V}'_k\mathbf{V}'_k})\cdot\hat{n}_k\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right]_{il}$$
$$+\left[\sum_{j=w,I}\int_{s_j}(-\bar{\delta}\bar{P}_k+\bar{\tau}_k-\overline{\rho_k\mathbf{V}'_k\mathbf{V}'_k})\cdot\hat{n}_k\frac{\mathrm{d}A_j}{\mathrm{d}z}\right]_{ij}+(\bar{\rho}_kA_k)_i\mathbf{g}\quad\text{for}\quad k=1,2 \qquad (26)$$

where w stands for a wall or an impenetrable fixed surface. In view of the applicability of boundary-layer approximation to the flow in the pin bundle for each phase, the normal components of stress tensor are small compared to the tangential components, the pressure is uniform over the cross section of a fluid and in view of equation (B5) is uniform over the whole cross section of a channel, i.e.

$$\bar{P}_k=\bar{P}_k(z,t)=\bar{P}(z,t).$$

Introducing these simplifications and condition of no slip at the wall, i.e. $\mathbf{V}_k=\mathbf{V}'_k=0$ at the wall, equation (26) becomes

$$\frac{\partial}{\partial t}(A_k\langle\langle\bar{\rho}_k\rangle\rangle\langle\tilde{V}_{kz}\rangle)_i+\frac{\partial}{\partial z}(A_k\langle\langle\bar{\rho}_k\tilde{V}^2_{kz}\rangle\rangle)_i$$
$$=-\frac{\partial}{\partial z}(A_k\bar{P})_i+\sum_l\left[\int_{s_k}(\bar{\tau}_z-\overline{\rho_k\mathbf{V}'_kV'_{kz}})\cdot\hat{n}_k\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right]_{il}$$
$$-\sum_l\left[\int_{s_k}(\bar{\rho}_k\tilde{\mathbf{V}}_k\tilde{V}_{kz})\cdot\hat{n}_k\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right]_{il}+\int_{s_w}\left(\bar{\tau}_{kzn}\frac{\mathrm{d}A_{kw}}{\mathrm{d}z}\right)_i+\int_{s_I}\Bigg[(-\bar{\delta}\bar{P}+\bar{\tau}_k-\overline{\rho_k\mathbf{V}'_k\mathbf{V}'_k})\cdot\hat{n}_k$$
$$-\bar{\rho}_k(\tilde{\mathbf{V}}_k-\mathbf{V}_I)\cdot\hat{n}_k\tilde{V}_{kz}\frac{\mathrm{d}A_I}{\mathrm{d}z}\Bigg]_i-\sum_l\left[\bar{P}\int_{s_k}(\hat{K}\cdot\hat{n}_k)\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right]_{il}-\bar{P}\int_{s_w}(\hat{K}\cdot\hat{n}_k)\frac{\mathrm{d}A_{kw}}{\mathrm{d}z}-(\bar{\rho}_kA_k)_ig_z. \qquad (27a)$$

The above equation can be simplified further by noting that

$$\int_{s_k}\hat{K}\cdot\hat{n}_k\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}=0 \qquad (27b)$$

and

$$\sum_{j=I,w}\int_{s_j}\hat{K}\cdot\hat{n}_k\frac{\mathrm{d}A_j}{\mathrm{d}z}=-\frac{\mathrm{d}A_k}{\mathrm{d}z}. \qquad (27c)$$

Equation (27a) becomes

$$\frac{\partial}{\partial t}(A_k\langle\langle\bar{\rho}_k\rangle\rangle\langle\tilde{V}_{kz}\rangle)_i+\frac{\partial}{\partial z}(A_k\langle\langle\bar{\rho}_k\tilde{V}^2_{kz}\rangle\rangle)_i$$
$$=-\frac{\partial}{\partial z}(A_k\bar{P})_i+\left(\bar{P}\frac{\partial A_k}{\partial z}\right)_i+\sum_l\left[\int_{s_k}(\bar{\tau}_k-\overline{\rho_k\mathbf{V}'_kV'_{kz}}-\rho_k\tilde{\mathbf{V}}_k\tilde{V}_{kz})\cdot\hat{n}_k\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right]_{il}$$
$$+\int_{s_{kw}}\left(\bar{\tau}_{kzn}\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right)_i+\int_{s_I}\left[(-\bar{\delta}\bar{P}+\bar{\tau}_k-\overline{\rho_k\mathbf{V}'_k\mathbf{V}'_k})\cdot\hat{n}_k-\bar{\rho}_k(\tilde{\mathbf{V}}_k-\mathbf{V}_I)\cdot\hat{n}_k\tilde{\mathbf{V}}_k\frac{\mathrm{d}A_I}{\mathrm{d}z}\right]_i$$
$$-(\bar{\rho}_kA_k)_ig_z\quad\text{for}\quad k=1,2. \qquad (28)$$

The momentum equation for mixture is obtained by adding momentum equation (28) for $k=1$ and 2 and using equation (B3). The result is

$$\frac{\partial}{\partial t}(A\rho_mV_m)+\frac{\partial}{\partial z}\{A[(1-\alpha)\langle\langle\bar{\rho}_1\tilde{V}^2_{1z}\rangle\rangle+\alpha\langle\langle\bar{\rho}_2\tilde{V}^2_{2z}\rangle\rangle]\}_i$$
$$=-A_i\frac{\partial\bar{P}_i}{\partial z}+\sum_{k=1}^{2}\sum_l\left[\int_{s_k}(\bar{\tau}_{kz}-\overline{\rho_k\mathbf{V}'_kV'_{kz}}-\bar{\rho}\tilde{\mathbf{V}}_k\tilde{V}_{kz})\cdot\hat{n}_k\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right]_{il}+\sum_{k=1}^{2}\int_{s_{kw}}\left(\bar{\tau}_{kzn}\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right)_i$$
$$-(\rho_mA)_ig_z. \qquad (29)$$

Using the definition

$$-s_w\tau_w=\sum_{k=1}^{2}\int_{s_k}\bar{\tau}_{kzn}\frac{\mathrm{d}A_{ke}}{\mathrm{d}z} \qquad (30)$$

the approximation,

$$\langle\!\langle \bar{\rho}_k \rangle\!\rangle \simeq \bar{\rho}_k \tag{31}$$

from which it implies that $\langle \Psi_k \rangle$ as defined by equation (12) becomes

$$\langle \Psi_k \rangle = \frac{\langle\!\langle \bar{\rho}_k \Psi_k \rangle\!\rangle}{\langle\!\langle \bar{\rho}_k \rangle\!\rangle} = \langle\!\langle \Psi_k \rangle\!\rangle \tag{32}$$

and introducing the correlation coefficient

$$C_k = \frac{\langle \tilde{V}_{kz}^2 \rangle}{(\langle \tilde{V}_{kz} \rangle)^2} \tag{33}$$

equation (29) becomes

$$\frac{\partial}{\partial t}(A\rho_m V_m)_i + \frac{\partial}{\partial z}\{A[(1-\alpha)\bar{\rho}_1 C_1(\langle \tilde{V}_{1z} \rangle)^2 + \alpha\bar{\rho}_2 C_2(\langle V_{2z} \rangle)^2]\}_i$$
$$= -A_i \frac{\partial \bar{P}_i}{\partial z} + \sum_{k=1}^{2} \sum_{l} \left[\int_{s_k} (\bar{\tau}_{kz} - \rho_k \overline{\mathbf{V}_k' V_{kz}'} - \bar{\rho}_k \tilde{\mathbf{V}}_k \tilde{V}_{kz}) \cdot \hat{n}_k \frac{\mathrm{d}A_{ke}}{\mathrm{d}z} \right]_{il} - (s_w \tau_w)_i - (\rho_m A)_i g_z. \tag{34}$$

Substituting for V_{1z} from equation (14) and (16b) as

$$\langle V_{1z} \rangle = V_m - \frac{\alpha}{1-\alpha} \frac{\bar{\rho}_2}{\bar{\rho}_1} V_{2m} \tag{35a}$$

and for V_{2z} from equation (14) as

$$\langle V_{2z} \rangle = V_m + V_{2m} \tag{35b}$$

$$\frac{\partial}{\partial t}(A\rho_m V_m)_i + \frac{\partial}{\partial z}\{A V_m[(1-\alpha)\bar{\rho}_1 C_1 \langle V_{1z} \rangle + \alpha\bar{\rho}_2 C_2 \langle V_{2z} \rangle]\}_i$$
$$+ \frac{\partial}{\partial z}\left\{ A\alpha\bar{\rho}_2 V_{2m} \left[(C_2 - C_1) V_m - \frac{C_2(1-\alpha)\bar{\rho}_1 + \alpha\bar{\rho}_2 C_1}{(1-\alpha)\bar{\rho}_1} V_{2m} \right] \right\}_i$$
$$= -A_i \frac{\partial \bar{P}_i}{\partial z} + \sum_{k=1}^{2} \sum_{l} \left[\int_{s_k} (\bar{\tau}_{kz} - \rho_k \overline{\mathbf{V}_k' V_{kz}'} - \bar{\rho}_k \tilde{\mathbf{V}}_k \tilde{V}_{kz}) \cdot \hat{n}_k \frac{\mathrm{d}A_{ke}}{\mathrm{d}z} \right]_{il} - (s_w \tau_w)_i - (\rho_m A)_i g_z. \tag{36}$$

Assuming $C_1 = C_2 \simeq 1$, i.e. approximately flat velocity profiles in both phases, and using equations (17) and (6b), equation (36) simplifies to

$$\frac{\partial}{\partial t}(A\rho_m V_m)_i + \frac{\partial}{\partial z}(A\rho_m V_m^2)_i$$
$$= -\frac{\partial}{\partial z}\left(\frac{Ac}{1-c} \rho_m V_{2m}^2 \right)_i - A_i \frac{\partial \bar{P}_i}{\partial z} + \sum_{k=1}^{2} \sum_{l} \left[\int_{s_k} (\bar{\tau}_{kz} - \rho_k \overline{\mathbf{V}_k' V_{kz}'}) \cdot \hat{n}_k \frac{\mathrm{d}A_{ke}}{\mathrm{d}z} \right.$$
$$\left. - \sum_{k=1}^{2} \sum_{l} (W_{ki} \tilde{V}_{kz})_l - (s_w \tau_w)_i - (\rho_m A)_i g_z. \right. \tag{37}$$

Momentum equation averaged over segment in y *direction*

In order to calculate diversion crossflow as defined by equation (6b) between subchannels we need to perform the averaging of the local momentum equation over a segment to obtain macroscopic momentum balance in transverse (to the interchannel gap) direction. For this purpose we utilize equations (A3) and (A4) and perform integration of equation (2) over a segment L_k parallel to y direction and require that the segment be bounded by A_k. We thus obtain

$$\frac{\partial}{\partial t}\left(\int_{L_k} \bar{\rho}_k \tilde{\mathbf{V}}_k \,\mathrm{d}y \right)_i + \frac{\partial}{\partial x}\left(\int_{L_k} \bar{\rho}_k \tilde{V}_{kx} \tilde{\mathbf{V}}_k \,\mathrm{d}y \right)_i + \frac{\partial}{\partial z}\left(\int_{L_k} \bar{\rho}_k \tilde{V}_{kz} \tilde{\mathbf{V}}_k \,\mathrm{d}y \right)_i$$
$$+ \left[\bar{\rho}_k (\tilde{\mathbf{V}}_k - \mathbf{V}_I) \cdot \hat{n}_k \tilde{\mathbf{V}}_k \frac{\mathrm{d}A_I}{\mathrm{d}x\,\mathrm{d}z} \right]_i + \sum_{L_k} \left[\bar{\rho}_k (\tilde{\mathbf{V}}_k \cdot \hat{n}_k) \tilde{\mathbf{V}}_k \frac{\mathrm{d}A_{ke}}{\mathrm{d}x\,\mathrm{d}z} \right]$$
$$= \frac{\partial}{\partial z}\left[\int_{L_k} (-\delta \bar{P} + \bar{\tau}_k - \rho_k \overline{\mathbf{V}_k' \mathbf{V}_k'}) \cdot \hat{K} \,\mathrm{d}y \right]_i + \frac{\partial}{\partial x}\left[\int_{L_k} (-\delta \bar{P} + \bar{\tau}_k - \rho_k \overline{\mathbf{V}_k' \mathbf{V}_k'}) \cdot \hat{i} \,\mathrm{d}y \right]_i$$
$$+ \sum_{L_k} \left[(-\delta \bar{P} + \bar{\tau}_k - \rho_k \overline{\mathbf{V}_k' \mathbf{V}_k'}) \cdot \hat{n}_k \frac{\mathrm{d}A_{ke}}{\mathrm{d}x\,\mathrm{d}z} \right]_i + \sum_{j=w,I} \left[(-\delta \bar{P} + \bar{\tau}_k - \rho_k \overline{\mathbf{V}_k' \mathbf{V}_k'}) \cdot \hat{n}_k \frac{\mathrm{d}A_j}{\mathrm{d}x\,\mathrm{d}z} \right]_i + (\bar{\rho}_k L_k)_i \mathbf{g} \tag{38}$$

where $\sum_{L_k}$ represents summation over all the extremities of the segment L_k.

Let us now consider the projection of the above equation over x-axis and apply boundary-layer approximations, which imply that we can neglect

$$\partial(\bar{\tau}_{kxx}-\overline{\rho_k V'_{kx} V'_{kx}})/\partial x,\ \partial(\bar{\tau}_{kzx}-\overline{\rho_k V'_{kz} V'_{kx}})/\partial z \text{ and } (\bar{\tau}_{kxx}-\overline{\rho_k V'_{kx} V'_{kx}}) \text{ (see [12])},$$

we obtain

$$\frac{\partial}{\partial t}\left(\int_{L_k}\bar{\rho}_k \tilde{V}_{kx}\,\mathrm{d}y\right)_i+\frac{\partial}{\partial x}\left(\int_{L_k}\bar{\rho}_k \tilde{V}_{kx}\tilde{V}_{kx}\,\mathrm{d}y\right)_i+\frac{\partial}{\partial z}\left(\int_{L_k}\bar{\rho}_k \tilde{V}_{kz}\tilde{V}_{kx}\,\mathrm{d}y\right)_i$$
$$=-\frac{\partial}{\partial x}\left(\int_{L_k}\bar{P}\,\mathrm{d}y\right)_i+\sum_{L_k}\left[(\bar{\tau}_{kxz}-\overline{\rho_k V'_{kz} V'_{kx}})\frac{\mathrm{d}A_{ke}}{\mathrm{d}x\,\mathrm{d}z}\right]-\sum_{L_k}\left[\bar{\rho}_k \tilde{V}_{kz}\tilde{V}_{kx}\frac{\mathrm{d}A_{ke}}{\mathrm{d}x\,\mathrm{d}z}\right]$$
$$+\bar{\tau}_{kxz}\frac{\mathrm{d}A_{kw}}{\mathrm{d}x\,\mathrm{d}z}+[(-\bar{\delta}\bar{P}+\bar{\tau}_k-\overline{\rho_k \mathbf{V}'_k \mathbf{V}'_k})\cdot\hat{n}_k-\bar{\rho}_k(\tilde{\mathbf{V}}_k-\mathbf{V}_I)\cdot\hat{n}_k\tilde{V}_{kx}]\frac{\mathrm{d}A_I}{\mathrm{d}x\,\mathrm{d}z}$$
$$+\bar{P}\frac{\mathrm{d}A_{kx}}{\mathrm{d}x\,\mathrm{d}z}-(\bar{\rho}_k L_k)_i g_x \quad \text{for} \quad k=1,2 \qquad (39)$$

where we have utilized equations (27b) and (27c) and A_{kx} is flow area for phase k normal to direction x.

To obtain momentum equation for the mixture we add momentum equation (39) for each phase and use equation (B3), the result is

$$\frac{\partial}{\partial t}\left(\sum_k\int_{L_k}\bar{\rho}_k \tilde{V}_{kx}\,\mathrm{d}y\right)_i+\frac{\partial}{\partial x}\left(\sum_k\int_{L_k}\bar{\rho}_k \tilde{V}_{kx}^2\,\mathrm{d}y\right)_i+\frac{\partial}{\partial z}\left(\sum_k\int_{L_k}\bar{\rho}_k \tilde{V}_{kz}\tilde{V}_{kx}\,\mathrm{d}y\right)_i$$
$$=-\left(L\frac{\partial\bar{P}}{\partial x}\right)_i+\sum_k\sum_{L_k}\left[(\bar{\tau}_{kxz}-\overline{\rho_k V'_{kz} V'_{kx}})\frac{\mathrm{d}A_{ke}}{\mathrm{d}x\,\mathrm{d}z}\right]-\sum_k\sum_{L_k}\left[\bar{\rho}_k \tilde{V}_{kz}\tilde{V}_{kx}\frac{\mathrm{d}A_{ke}}{\mathrm{d}x\,\mathrm{d}z}\right]$$
$$+\left(\bar{\tau}_{xz}\frac{\mathrm{d}A_{kw}}{\mathrm{d}x\,\mathrm{d}z}\right)_i-\sum_k(\bar{\rho}_k L_k)_i g_x. \qquad (40)$$

Orientating z axis along vertical direction so that $g_x=0$, applying equation (40) at interface (or gap) between two connected channels and identifying with the use of equation (6b) that

$$\int_{L_k}\bar{\rho}_k \tilde{V}_{kx}\,\mathrm{d}y=\left(\int_{s_k}\bar{\rho}_k \mathbf{V}_k\cdot\hat{n}_k\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right)_{il}=W_{kil} \qquad (41a)$$

and assuming

$$(\tilde{V}_{kx})_i=[\tilde{V}_{kx}(x,t)]_i \qquad (41b)$$

so that

$$\int_{L_k}\bar{\rho}_k \tilde{V}_{kx}\tilde{V}_{kx}\,\mathrm{d}y=W_{kil}\tilde{V}_{kx}^*=W_{kil}^2/\bar{\rho}_k^* s_k \qquad (41c)$$

$$\int_{L_k}\bar{\rho}_k \tilde{V}_{kx}\tilde{V}_{kz}\,\mathrm{d}y\simeq W_{kil}\tilde{V}_{kz}^* \qquad (41d)$$

where $\bar{\rho}_k^*$ and V_{kz}^* are the average values of density and axial component of velocity for phase k, respectively, at the gap between two channels. In view of equations (41), equation (40) becomes

$$\frac{\partial}{\partial t}W_{il}+\frac{\partial}{\partial x}\left(\sum_k W_{kil}^2/\bar{\rho}_k^* s_k\right)+\frac{\partial}{\partial z}\sum_k W_{kil}\tilde{V}_{kz}^*$$
$$=-\left(s\frac{\partial P}{\partial x}\right)_{il}+\left(\bar{\tau}_{xz}\frac{\mathrm{d}A_w}{\mathrm{d}x\,\mathrm{d}z}\right)_{il}+\sum_k\left[(\bar{\tau}_{kxz}-\overline{\rho_k V'_{kz} V'_{kz}}-\bar{\rho}_k\tilde{V}_{kz}\tilde{V}_{kx})\frac{\mathrm{d}A_{ke}}{\mathrm{d}x\,\mathrm{d}z}\right]_{il} \qquad (42)$$

Energy equation averaged over cross section

The use of equations (A1) and (A2) to equation (3) yields

$$\frac{\partial}{\partial t}(A_k\langle\langle\bar{\rho}_k\tilde{E}_k\rangle\rangle)_i+\frac{\partial}{\partial z}(A_k\langle\langle\bar{\rho}_k\tilde{E}_k\tilde{V}_{kz}\rangle\rangle)_i$$
$$+\left[\int_{s_I}\bar{\rho}_k\tilde{E}_k(\tilde{\mathbf{V}}_k-\mathbf{V}_I)\cdot\hat{n}_k\frac{\mathrm{d}A_I}{\mathrm{d}z}\right]_i+\sum_l\left[\int_{s_k}\bar{\rho}_k\tilde{E}_k\tilde{\mathbf{V}}_k\cdot\hat{n}_k\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right]_{il}$$
$$=\frac{\partial}{\partial z}[A_k\langle\langle(-\bar{\mathbf{q}}_k-\overline{\rho_k E'_k\mathbf{V}'_k}-\overline{P_k\mathbf{V}_k}+\overline{\tau_k\cdot\mathbf{V}_k})\cdot\hat{K}\rangle\rangle]_i+\left[\int_{s_I}(-\bar{\mathbf{q}}_k-\overline{\rho_k E'_k\mathbf{V}'_k}+\overline{\tau_k\cdot\mathbf{V}_k}-\overline{P_k\mathbf{V}_k})\cdot\hat{n}_k\frac{\mathrm{d}A_I}{\mathrm{d}z}\right]_i$$

$$+\sum_{l}\left[\int_{s_k}(-\bar{\mathbf{q}}_k-\overline{\rho_k E'_k \mathbf{V}'_k}-\overline{P_k \mathbf{V}_k}+\overline{\tau_k\cdot\mathbf{V}_k})\cdot\hat{n}_k\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right]_{il}$$
$$+\left[\int_{s_{kw}}(-\bar{\mathbf{q}}_k-\overline{\rho_k E'_k \mathbf{V}'_k}-\overline{P_k \mathbf{V}_k}+\overline{\tau_k\cdot\mathbf{V}_k})\cdot\hat{n}_k\frac{\mathrm{d}A_{kw}}{\mathrm{d}z}\right]+(\langle\!\langle\bar{\rho}_k\mathbf{g}\cdot\mathbf{V}_k\rangle\!\rangle A_k)_i. \quad (43)$$

Since τ_{kzz} is small in comparison with P_k, and since $\tilde{\mathbf{V}}_{ky}$ and $\tilde{V}_{kx}$ are small in comparison with $\tilde{V}_{kz}$, we neglect the term

$$\frac{\partial}{\partial z}[A_k\langle\!\langle\overline{\tau_{kzz}V_{kz}}+\overline{\tau_{kzy}V_{ky}}+\overline{\tau_{kzx}V_{kx}}\rangle\!\rangle]_i$$

in comparison with

$$\frac{\partial}{\partial z}[\langle\!\langle-\overline{P_k V_{kz}}\rangle\!\rangle]_i.$$

In view of no slip condition at the wall

$$-\overline{\rho_k E'_k V'_k} = -\overline{P_k \mathbf{V}_k} = \overline{\tau_k\cdot\mathbf{V}_k} = 0.$$

Since we are dealing with low speed flows, we can therefore neglect pressure work and dissipation work at the gap between the two channels in comparison with $-\bar{\mathbf{q}}_k-\overline{\rho_k E'_k \mathbf{V}'_k}$.

In view of the above discussed approximation, equation (43) becomes

$$\frac{\partial}{\partial t}(A_k\langle\!\langle\bar{\rho}_k E_k\rangle\!\rangle)_i+\frac{\partial}{\partial z}(A_k\langle\!\langle\bar{\rho}_k\tilde{E}_k\,\tilde{V}_{kz}\rangle\!\rangle)_i$$
$$=\frac{\partial}{\partial z}[A_k\langle\!\langle(-\bar{\mathbf{q}}_k-\overline{\rho_k E'_k\mathbf{V}'_k}-\overline{P_k\mathbf{V}_k})\cdot\hat{K}\rangle\!\rangle]_i$$
$$-\left\{\int_{s_I}[\bar{\rho}_k\tilde{E}_k(\tilde{\mathbf{V}}_k-\mathbf{V}_I)+\bar{\mathbf{q}}_k+\overline{\rho_k E'_k\mathbf{V}'_k}-\overline{\tau_k\cdot\mathbf{V}_k}+\overline{P_k\mathbf{V}_k}]\cdot\hat{n}_k\frac{\mathrm{d}A_I}{\mathrm{d}z}\right\}_i$$
$$-\sum_l\left[\int_{s_k}(\bar{\rho}_k\tilde{E}_k\mathbf{V}_k+\bar{\mathbf{q}}_k+\overline{\rho_k E'_k\mathbf{V}'_k})\cdot\hat{n}_k\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right]_l-\left[\int_{s_{kw}}\bar{\mathbf{q}}_k\cdot\hat{n}_k\frac{\mathrm{d}A_{kw}}{\mathrm{d}z}\right]_i$$
$$-(\langle\!\langle\bar{\rho}_k g_z V_{kz}\rangle\!\rangle A_k)_i. \quad (44)$$

Adding energy equation (44) for each phase and using equations (B7), (27), (31), (32), and (35) we obtain

$$\frac{\partial}{\partial t}(A\rho_m E_m)_i+\frac{\partial}{\partial z}[(A_1\bar{\rho}_1\gamma_1\langle\tilde{E}_1\rangle+A_2\bar{\rho}_2\gamma_2\langle\tilde{E}_2\rangle)V_m]_i$$
$$=-\frac{\partial}{\partial z}\left[\left(A_2\bar{\rho}_2\gamma_2\langle\tilde{E}_2\rangle-A_1\gamma_1\langle\tilde{E}_1\rangle\frac{\alpha\bar{\rho}_2}{1-\alpha}\right)V_{2m}\right]_i+\frac{\partial}{\partial z}\left[\sum_k A_k\langle\!\langle(\bar{q}_{kz}-\overline{\rho_k E'_k V'_{kz}}-\overline{PV_{kz}})\rangle\!\rangle\right]_i$$
$$-\sum_k\sum_l\left[\int_{s_k}\rho_k\tilde{E}_k\tilde{V}_{kx}+\bar{q}_{kx}+\overline{\rho_k E'_k V'_{kx}}\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right]_{il}+(q_w s_w)_i-(\rho_m V_m A)_i g_z \quad (45)$$

where

$$-q_w s_w=\sum_k\int_{s_{kw}}\bar{\mathbf{q}}_k\cdot\hat{n}_k\frac{\mathrm{d}A_{kw}}{\mathrm{d}z} \quad (46a)$$

$$\gamma_k=\frac{\langle\tilde{E}_k\tilde{V}_{kz}\rangle}{\langle\tilde{E}_k\rangle\langle\tilde{V}_{kz}\rangle} \quad (46b)$$

$$E_m=(\bar{\rho}_1(1-\alpha)\langle\tilde{E}_1\rangle+\bar{\rho}_2\alpha\langle\tilde{E}_2\rangle)/\rho_m. \quad (46c)$$

For the case of adiabatic flow of bulk boiling such as due to sudden depressurization in case of water reactor, and in the case of liquid metal boiling, the total internal energies of each phase are essentially uniform across the cross section of a channel. Thus it follows for these cases that $\gamma_k \simeq 1$; however, in the presence of substantial degree of thermodynamic non-equilibrium the coefficients γ_k are lkiely to be different from unity. With the substitution, $\gamma_k \simeq 1$, equation (45) becomes

$$\frac{\partial}{\partial t}(A\rho_m E_m)_i+\frac{\partial}{\partial z}(A\rho_m E_m V_m)_i$$
$$=-\frac{\partial}{\partial z}(A\alpha\bar{\rho}_2\Delta E_{fg}V_{2m})_i+\frac{\partial}{\partial z}\left[\sum_k A_k\langle\!\langle(-\bar{q}_{kz}-\overline{\rho_k E'_k V'_{kz}}-\overline{\bar{P}\,\tilde{V}_{kz}})\rangle\!\rangle\right]_i$$
$$-\sum_k\sum_l\left[\int_{s_k}(\rho_k\tilde{E}_k\tilde{V}_{kx}+\bar{q}_{kx}+\overline{\rho_k E'_k V'_{kx}})\frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right]_{il}+(q_w S_w)_i-(\rho_m V_m A)_i g_z \quad (47)$$

where

$$\Delta E_{fg} = \langle \tilde{E}_2 \rangle - \langle \tilde{E}_1 \rangle. \tag{48}$$

In arriving at equation (47), we have made the following simplifications:

$$\overline{PV_{kz}} = (\bar{P}+P'')(\bar{V}_{kz}+V''_{kz}) = \overline{PV_{kz}} + \overline{P''V''_{kz}},$$

at moderate Mark numbers it can easily be shown that (see e.g. [13])

$$\bar{V}_{kz} \simeq \tilde{V}_{kz}$$

and that $\overline{P''V''_{kz}} + \overline{PV_{kz}} \simeq \bar{P}\tilde{V}_{kz}$.

Sometimes it is more convenient to express the energy equation in terms of static enthalpy $H_k = U_k + P_k/\rho_k$, rather than total internal energy E_k. Then equation (47) becomes

$$\begin{aligned}\frac{\partial}{\partial t}(A\rho_m H_m)_i + \frac{\partial}{\partial z}(A\rho_m H_m V_m)_i &= -\frac{\partial}{\partial z}(A\alpha\bar{\rho}_2 \Delta H_{fg} V_{2m})_i + \frac{\partial}{\partial z}\left[\sum_k A_k \langle\langle(-\bar{q}_{kz} - \overline{\rho_k H'_k V'_{kz}})\rangle\rangle\right]_i \\ &\quad - \sum_k \sum_l \left[\int_{s_k} (\bar{\rho}_k \tilde{H}_k \tilde{V}_{kx} + \bar{q}_{kx} + \overline{\rho_k H'_k V'_{kx}}) \frac{\mathrm{d}A_{ke}}{\mathrm{d}z}\right]_{il} \\ &\quad + (q_w s_w)_i + \frac{\partial}{\partial t}(AP)_i + \left[A\left(V_m + \frac{\alpha\Delta\rho}{\bar{\rho}_1} V_{2m}\right)\frac{\partial P}{\partial z}\right]_i + (V_m \tau_w s_w)_i.\end{aligned} \tag{49}$$

COMPARISON WITH PREVIOUS STUDIES AND CONCLUDING REMARKS

Among the previous two-phase models for a pin bundle, the one by Rowe [1, 2] has enjoyed a greater degree of success because of the versatile nature of the original coding for single phase flows. Rowe extended Meyer's [14] derivation of equations of motion for one-dimensional two-phase flows to a pin bundle geometry. Meyer's model assumes thermal equilibrium between the liquid and vapor phases either at a local pressure or at the mean system pressure level. A good discussion of this deficiency in the model in relation to its capabilities for analyses of two-phase flows during accident conditions such as loss-of-flow due to sudden depressurization, can be found in a paper by Zuber and Dougherty [15]. In view of this assumption, the need for additional continuity equation (25) was eliminated. We may note here that the net vaporization term containing Γ_2 plays the dominant role in determining the degree of thermal non-equilibrium of the mixture and in fact the degree of thermal non-equilibrium cannot be determined without the inclusion of the second continuity equation. Meyer's model is based on slip ratio rather than center of mass or volume model as utilized here. Consequently, Meyer found it necessary to introduce four separate definitions of mixture specific volumes i.e. an area averaged specific volume, a momentum averaged specific volume, a kinetic energy averaged specific volume, and a velocity weighted specific volume; each of these four specific volumes exhibit a different functional dependence on the slip ratio. Consequently, as discussed by Zuber and Dougherty [15] there can arise serious discrepancies in the prediction of critical two-phase flows if such a model is utilized.

Another point of departure in previous models [1–4] from the present analysis is the derivation of momentum equation for cross flow. The form of this equation utilized in these computer codes has evolved over several years. The derivation of this equation in its earlier crude form (see for example, [1]) merely consisted of two terms, namely, the transverse pressure gradient being proportional to the second power of the diversion crossflow. All effects of acceleration were considered unimportant and therefore disregarded. The present form of this equation as for example utilized in COBRA-IV [16] represents a considerable improvement over its earlier crude form, however its derivation is based on a mere intuitive adaptation of local transverse momentum balance by neglecting terms contributed by flow in y direction and the terms considered intuitively to be unimportant. Of course, this equation can only be correctly formulated by carrying out a segment averaging of the local transverse momentum balance, where the segment is taken parallel to y direction. A term by term comparison of our governing equation with those by others is not fruitful in view of the degree of uncertainty and empiricism involved in the present derivation of constitutive relationships for various crossflow terms; as a result the importance of individual terms retained in our analysis at present cannot be assessed.

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APPENDIX A

The Leibnitz's theorem over surfaces

Let us consider a volume V_k contained between two parallel planes cutting V_k along two cross sections $A_k|_z$ and $A_k|_{z+\Delta z}$ infinitely close together with distance Δz between them. The Leibnitz's theorem valid over surfaces is given as [9]

$$\iint_{A_k} \frac{\partial \Psi_k}{\partial t}\,\mathrm{d}A = \frac{\partial}{\partial t}(A_k \langle\langle \Psi_k \rangle\rangle) - \sum_{l=e,I} \int_{s_{kl}} (\mathbf{V}_l \cdot \hat{n}_k)\Psi_k \frac{\mathrm{d}A_{kl}}{\mathrm{d}z} \quad \text{(A1)}$$

where s_{kl} is a wetted perimeter at the boundary (e.g. interface, external surfaces) and A_{kl} is the interfacial area.

The divergence theorem over surfaces

The divergence theorem valid for surface is given as [9]

$$\iint_{A_k} \nabla \cdot \mathbf{F}_k\,\mathrm{d}A = \frac{\partial}{\partial z}(A_k \langle\langle F_{kz} \rangle\rangle) + \sum_{l=e,I} \int_{s_{kl}} \mathbf{F}_k \cdot \hat{n}_k \frac{\mathrm{d}A_{kl}}{\mathrm{d}z}. \quad \text{(A2)}$$

The Leibnitz's theorem over a segment

Let us consider a volume V along two sections S_1 and S_2 perpendicular to 0x distance Δx apart. The Leibnitz's theorem for segment L_k parallel to direction y and bounded by A_k, is given as [12]

$$\int_{L_k} \frac{\partial \Psi_k}{\partial t}\,\mathrm{d}y = \frac{\partial}{\partial t}\int_{L_k} \Psi_k\,\mathrm{d}y - \sum_{l=e,I} (\mathbf{V}_l \cdot \hat{n}_k)\Psi_k \frac{\mathrm{d}A_{kl}}{\mathrm{d}x\,\mathrm{d}z} \quad \text{(A3)}$$

where summation represents sum over all extremities of the segment L_k, interfaces and walls included.

The divergence theorem over segment

The divergence theorem valid over segment L_k can be stated as [12]

$$\int_{L_k} \nabla \cdot \mathbf{F}_k\,\mathrm{d}y = \frac{\partial}{\partial z}\int_{L_k} F_{kz}\,\mathrm{d}y + \frac{\partial}{\partial x}\int_{L_k} F_{kx}\,\mathrm{d}y + \sum_{l=e,I} \mathbf{F}_k \cdot \hat{n}_k \frac{\mathrm{d}A_{kl}}{\mathrm{d}x\,\mathrm{d}z}. \quad \text{(A4)}$$

APPENDIX B

Interfacial balance conditions

Since an interface between the phases is a singular surface across which the fluid density, energy and velocity suffer jump discontinuities, therefore the standard differential balance equations valid separately for each phase cannot be applied across the interface. In order to take into account the singular characteristics i.e. discontinuities in various variables, jump conditions consisting of a form of balance equations are developed. A detail discussion of the method for derivation of the jump conditions can be found in Ishii [11], Delhaye [17] and Kocamustafaogullari [9]. For the present application we merely state these interface balance conditions.

Mass balance at interface.

$$\sum_{k=1}^{2} \bar{\rho}_k(\tilde{\mathbf{V}}_k - \mathbf{V}_I)\cdot \hat{n}_k = 0. \quad \text{(B1)}$$

Condition of no slip at the interface.

$$\tilde{V}_{1t} = \tilde{V}_{2t} \quad \text{(B2)}$$

where $\tilde{V}_{kt}$ is the tangential (to the interface) component of the fluid velocity $\tilde{\mathbf{V}}_k$.

Momentum balance at the interface.

$$\sum_{k=1}^{2} \{\bar{\rho}_k(\tilde{\mathbf{V}}_k - \mathbf{V}_I)\cdot \hat{n}_k \tilde{\mathbf{V}}_k + [\bar{P}_k \bar{\bar{\delta}} - (\bar{\bar{\tau}}_k - \rho_k \overline{\mathbf{V}'_k \mathbf{V}'_k})]\cdot \hat{n}_k\} = 0 \quad \text{(B3)}$$

where $\bar{\bar{\delta}}$ is the unit tensor. Here, we have neglected the stress tensor due to interfacial tension between the two phases. This assumption is valid provided that both the curvature of the interface and the surface tension gradient along the interface are small. If we neglect the initial growth of the bubble, the curvature in the subsequent two-phase motion can be neglected. Surface tension gradient along the interface will be negligible provided surfactant effects are absent.

Since the momentum balance equation (B3) is a vector equation, we must therefore obtain normal and tangential components of it.

Normal component. To obtain the normal component we take the dot product of equation (B3) with $\hat{n}_1$, and utilize the fact that $\hat{n}_1 = -\hat{n}_2$ and equation (B1), thus resulting in

$$\dot{m}_1(\tilde{\mathbf{V}}_1 - \tilde{\mathbf{V}}_2)\cdot \hat{n}_1 + (\bar{P}_1 - \bar{P}_2) + (\bar{\tau}_{2nn} - \rho_2 \overline{V'^2_{2n}}) - (\bar{\tau}_{1nn} - \rho_1 \overline{V'^2_{1n}}) = 0 \quad \text{(B4)}$$

where $\dot{m}_k = \bar{\rho}_k(\tilde{\mathbf{V}}_k - \mathbf{V}_I)\cdot \hat{n}_k$. Vernier and Delhaye [12] have demonstrated by order-of-magnitude analysis that contribution to normal stress due to interfacial mass transfer (first term in the above equation) is negligibly small. By utilizing Prandtl's boundary-layer approximations (which are applicable in view of the fact that the characteristic axial length scale such as the length of the pin bundle, is considerably larger than the characteristic length scale in the transverse direction), we can easily demonstrate normal stresses at the interface are of second order. Consequently, equation (B4) simplifies to

$$\bar{P}_1 = \bar{P}_2. \tag{B5}$$

Tangential component. Taking the tangential component of equation (B3), we obtain

$$\bar{\tau}_{1nt} - \overline{\rho_1 V'_{1n} V'_{1t}} = \bar{\tau}_{2nt} - \overline{\rho_2 V'_{2n} V'_{2t}}. \tag{B6}$$

Energy balance at the interface.

$$\sum_{k=1}^{2} \{\dot{m}_k \tilde{E}_k + [\bar{\mathbf{q}}_k - (\overline{\tau_k \cdot \mathbf{V}_k} - \overline{\rho_k E'_k \mathbf{V}'_k} - \overline{P_k \mathbf{V}_k})] \cdot \hat{n}_k\} = 0. \tag{B7}$$

EQUATIONS DU MOUVEMENT D'UN FLUIDE DIPHASIQUE DANS UNE GRAPPE D'AIGUILLES D'UN REACTEUR NUCLEAIRE

Résumé—Les bilans macroscopiques sur le modèle à deux fluides dans une grappe d'aiguilles sont obtenus en considérant la moyenne eulerienne, dans la section du canal, des équations de continuité, de quantité de mouvement et d'énergie pour un écoulement turbulent à une seule phase, en supposant que chaque phase, dans l'écoulement diphasique, est un milieu continu couplé par des conditions appropriées "de saut" à l'interface. Pour déterminer l'écoulement transversal, une équation de quantité de mouvement dans la direction transversale est obtenue pour chaque phase en considérant la moyenne eulerienne de l'équation locale de quantité de mouvement, le segment étant parallèle à la distance entre les aiguilles. Considérant le mélange comme un tout, on formule un modèle de diffusion basé sur la vitesse massique d'entrainement. Dans la direction axiale, il est exprimé en fonction des trois équations de conservation de la masse, de la quantité de mouvement et de l'énergie pour le mélange avec une équation additionnelle pour la phase vapeur. Pour la détermination du mouvement secondaire, on obtient une équation de quantité de mouvement transverse pour le mélange. On montre que la formulation antérieure d'un modèle diphasique, basé sur le modèle d'écoulement avec glissement et des bilans intégraux de sous-canaux avec des volumes de contrôle finis, est inadéquat pour les raisons suivantes: (a) Le modèle est heuristique et il suppose a priori l'ordre de grandeur des termes. Un excellent exemple est fourni par la manière dont on établit l'équation de quantité de mouvement transversal depuis la forme première. (b) Le modèle est incomplet et incorrect quand on l'applique à des mélanges diphasiques en non équilibre thermique comme dans le cas d'une dépressurisation accidentelle d'un réacteur à refroidissement par l'eau.

BEWEGUNGSGLEICHUNGEN FÜR DIE ZWEI-PHASENSTRÖMUNG IN EINEM BRENNSTAB-BÜNDEL EINES KERNREAKTORS

Zusammenfassung—Für turbulente Ein-Phasenströmung wird auf die lokalen Kontinuitäta-, Impuls- und Energiegleichungen über einen Kanalquerschnitt die Flächenmittelung nach Euler angewandt. Hierfür wird angenommen, daß bei Zwei-Phasenströmungen jede Phase für sich stetig ist, jedoch an der Phasengrenze durch geeignete Sprung-Bedingungen an die andere gekoppelt wird. Damit ergeben sich entsprechende makroskopische Gleichgewichts-Beziehungen in axialer Richtung für ein Zwei-Stoff-Modell in einem Brennstab-Bündel. Zur Berechnung der Querströmung erhält man für jede Phase eine Impulsgleichung quer zum Spalt zwischen den Stäben, indem man die Segment-Mittelung nach Euler auf die lokale Impulsgleichung anwendet. Das Segment wird hierbei parallel zum Spalt gewählt. Indem man das Gemisch als Ganzes betrachtet, wird auf der Grundlage der Driftstromgeschwindigkeit ein Diffusionsmodell erstellt. In axialer Richtung wird dies in Form der drei auf das Gemisch angewandten Bilanzgleichungen für Masse, Impuls und Energie mit einer zusätzlichen Kontinuitätsgleichung für die Dampf-Phase ausgedrückt. Zur Berechnung der Querströmung wird für ein Gemisch die Quer-Impuls-Gleichung aufgestellt. Es wird erörtert, daß der frühere Ansatz für die Zwei-Phasenströmung, der auf einem "Gleitstrom"-Modell und einer Bilanzbildung über sämtliche Teilkanäle unter Benutzung finiter Kontroll-Volumina beruht in folgender Hinsicht unzulänglich ist: (a) Das Modell ist heuristisch und enthält à priori-Annahmen über die Größenordnung der Ausdrücke. Ein hervorragendes Beispiel hierfür gibt die Art und Weise, in der sich die Quer-Impuls-Gleichung seit ihrer ersten groben Formulierung entwickelt hat. (b) Das Modell ist unvollständig und unrichtig, wenn es auf Zwei-Phasen-Gemische angewandt wird, die nicht im thermischen Gleichgewicht stehen, wie zum Beispiel während einer unfallbedingten Druckentlastung eines wassergekühlten Reaktors.

УРАВНЕНИЯ ДВИЖЕНИЯ ДЛЯ ДВУХФАЗНОГО ПОТОКА ПРИ ОБТЕКАНИИ ПУЧКА ТОНКИХ СТЕРЖНЕЙ ЯДЕРНОГО РЕАКТОРА

Аннотация — С помощью эйлеровского усреднения по площади канала уравнений сохранения массы, количества движения и энергии для однофазного турбулентного потока в предположении, что каждая из фаз двухфазного потока является непрерывной, но связанной условиями «скачка» на границе раздела, получены соответствующие аксиальные макроскопические балансные уравнения для модели двух жидкостей, обтекающих пучок тонких стержней. Для определения поперечного потока получено уравнение количества движения в поперечном (по отношению к зазору между стержнями) направлении для каждой фазы с использованием эйлеровского усреднения; при этом сегмент выбирается параллельным зазору. Для смеси в целом сформулирована диффузионная модель, основанная на скорости дрейфа потока. В аксиальном направлении она представлена тремя уравнениями сохранения массы, количества движения и энергии смеси и дополнительным уравнением неразрывности для паровой фазы. Поперечный поток определяется из уравнения для поперечного количества движения смеси. Показано, что ранее предложенное описание двухфазного потока на основании модели «скольжения» потока и интегральных балансовых соотношений с использованием конечных контрольных объёмов непригодно в силу следующих причин: (а) модель является эвристической и даёт априорное определение порядков величин. Характерным примером служит метод получения уравнения для поперечного количества движения; (б) модель является неполной и неверной в приложении к двухфазным смесям при наличии теплового неравновесия, как, например, в случае мгновенного падения давления в реакторе с водяным охлаждением.